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Dipole–quadrupole decomposition of two coupled spin 1 systems

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Abstract

Using dipole and quadrupole operators as the orthogonal basis of $su(3)$ algebra and successive Cartan decompositions, the decomposition of matrices for two coupled spin 1 systems is investigated so as to meet the requirements of some realistic quantum systems. Finally, this kind of decomposition for the ternary SWAP gate is given specifically. This method can be used to investigate the realization of two-qutrit logic gates.

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1. Introduction

Recent studies have indicated that there are many advantages to expanding quantum computers from double-valued systems (qubits) to multi-valued systems (qudits) [1–3]. There have also been many proposals to use multi-valued systems in quantum cryptography [4, 5], quantum teleportation [6] and quantum computation [1–3, 7–11]. Three-level quantum systems, so-called qutrits, are the simplest multi-valued systems. Although much work has been done in the proposals for the quantum information processes of multi-valued systems and the synthesis of multi-valued quantum logic circuits, the work in implementation and optimization of multi-valued quantum gates is relatively less.

Decomposition of the matrix plays a very important role in implementing and optimizing quantum gates. The decomposition methods currently used are mainly Cartan decomposition based on group theory [12], Cosine–Sine decomposition (CSD) [13] based on numerical linear algebra, and the quantum Shannon decomposition (QSD) [14] proposed by Shende, Bullock and Markov. The most widely used matrix decomposition in quantum information science is a decomposition of the $SU(2^n)$ group for the n-qubit systems [15–18]. In [10] and [11], the use of CSD, the realization of multi-qutrit and multi-qudit gates is discussed. A kind of Cartan decomposition for the bipartite quantum system in high dimension is given in [19], and the Cartan decomposition for two-qutrit systems is discussed in [20]. Based on this decomposition, the realization of the two-qutrit logic gate in a bipartite three-level system with the quasi-Ising interaction is investigated and the realization of the ternary SWAP gate and the ternary $\sqrt{\mathrm{SWAP}}$ gate is discussed specifically in [21].

Spin 1's are a very common example of three-level systems. Examples are the nuclear spins of naturally occurring isotopes ^{2}H, ^{6}Li, ^{14}N and a long-life radioactive isotope ^{32}P. In addition, the spin-1 systems in condensate and other research areas have attracted much attention [22–25], and any three-level system can also be considered as quasi-spin 1's. The interaction Hamiltonian to couple spin 1's can be constructed by the dipole and quadrupole operators described below. The simplest one is the one parameter Ising interaction [26]. The Heisenberg interaction is an interaction that often appears in literature [27–30]. The controllability of two spin 1's coupled with that interaction has been discussed in [30]. It may have a quadrupole interaction [24, 25, 31]. The implementation of unitary operations in a one-qutrit system is discussed in [31], and the quadrupole degree of freedom plays an important role for that. In fact, as we shall see, the quadratic term in the Hamiltonian is a quadrupole term.

A two-qutrit logic gate can be resolved into four 1-qutrit quantum multiplexers and three 1-qutrit uniformly controlled rotations acting on the first qutrit [10]. But the realization of these two basic components needs further study. The bases of the Cartan decomposition for bipartite quantum systems in [19] are pure mathematical, and the bases in [20] are different from those in [19], but they still cannot meet the requirements of some realistic systems.

In this paper, using dipole and quadrupole operators as the orthogonal basis of the $su(3)$ algebra, we carry out specific Cartan decompositions of matrices for two coupled spin 1 systems. This paper is organized as follows. In the second section, we recall the definition of Cartan decomposition, and then we give the dipole-quadrupole bases of the $u(3)$ algebra in section 3. The dipole–quadrupole decomposition of two coupled spin 1's is investigated in section 4. The realization of the two-qutrit gate is discussed briefly and an example to decompose the ternary SWAP gate is given in section 5. The conclusion is given in section 6.

2. Cartan decomposition of the Lie group

The Cartan decomposition of the Lie group depends on the decomposition of its Lie algebras. Let g be a semisimple Lie algebra and there is a decomposition

$$g = l \oplus p, \tag{1}$$

where l and p satisfy the commutation relations

$$[l, l] \subseteq l, \qquad [l, p] \subseteq p, \qquad [p, p] \subseteq l; \tag{2}$$

we said that the decomposition is the Cartan decomposition of the Lie algebra g. The l is closed under the Lie bracket, so it is a Lie subalgebra of g, and that $p = l^{\perp}$. A maximal Abelian subalgebra a contained in p is called a Cartan subalgebra, and the dimension of a is called the rank of the decomposition. Then, utilizing the relation between the Lie group and the Lie algebra, for every element X of the Lie group G can be written as

$$X = K_1 A K_2, \tag{3}$$

where $G = e^g$, K_1, $K_2 \in e^l$ and $A \in e^a$. The coset space G/e^l is called a Riemannian symmetric space of G.

The all Riemannian symmetric space of classical groups can be classified into several types [12], and up to conjugacy, the corresponding decompositions fall into one of few types. Same as [19, 20], we shall only use the decompositions of AI type for the $SU(n)$ group (the subgroup is isomorphism to the $SO(n)$ group) and decompositions of BDI type for the $SO(n)$ group (the subgroup is isomorphism to $SO(d_1) \oplus SO(d_2)$, $d_1 + d_2 = n$).

Table 1. The commutation relations between $\mathcal{Q}$'s.

[,]	U_2	V_1	V_2	Q_0
U_1	$2\mathrm{i}L_z$	$-\mathrm{i}L_y$	$-\mathrm{i}L_x$	0
U_2		$\mathrm{i}L_x$	$-\mathrm{i}L_y$	0
V_1			$\mathrm{i}L_z$	$-\sqrt{3}\mathrm{i}L_y$
V_2				$\sqrt{3}\mathrm{i}L_x$

Table 2. The commutation relations between $\mathcal{L}$ and $\mathcal{Q}$.

[,]	U_1	U_2	V_1	V_2	Q_0
L_x	$-\mathrm{i}V_2$	$\mathrm{i}V_1$	$-\mathrm{i}U_2$	$\mathrm{i}(U_1+\sqrt{3}Q_0)$	$-\sqrt{3}\mathrm{i}V_2$
L_y	$-\mathrm{i}V_1$	$-\mathrm{i}V_2$	$\mathrm{i}(U_1-\sqrt{3}Q_0)$	$\mathrm{i}U_2$	$\sqrt{3}\mathrm{i}V_1$
L_z	$2\mathrm{i}U_2$	$-2\mathrm{i}U_1$	$\mathrm{i}V_2$	$-\mathrm{i}V_1$	0

3. Dipole-quadrupole bases of the $u(3)$ algebra

The matrices of a one-qutrit gate are elements of the $SU(3)$ group. To meet the requirement of spin 1's particles with the specific interaction, we take dipole and quadrupole operators as the orthogonal basis of the $su(3)$ algebra. The matrices of three dipole operators are

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

The five quadrupole operators are

$$\begin{aligned} U_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \\ V_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}, \qquad Q_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{5}$$

Multiplying these eight Hermitian matrices by i, we gain the basis vectors of the Lie algebra $su(3)$. The basis is similar to but different from that in [18]. Together with the 3×3 identity matrix multiplied by i, they constitute the basis vectors of the Lie algebra $u(3)$.

The dipole operators just are angular momentum operators and satisfy the commutation relation

$$[L_\alpha, L_\beta] = \mathrm{i}L_\gamma (\alpha, \beta, \gamma \to x, y, z). \tag{6}$$

We denote the set of dipole operators by $\mathcal{L}$ and quadrupole operators by $\mathcal{Q}$. The set commutation relations between $\mathcal{Q}$'s and between $\mathcal{L}$ and $\mathcal{Q}$ are given in tables 1 and 2 respectively.

Let

$$L_\pm = L_x \pm \mathrm{i}L_y, \qquad L_0 = L_z, \qquad Q_{\pm 1} = \mp\tfrac{1}{\sqrt{2}}(V_1 \pm \mathrm{i}V_2), \qquad Q_{\pm 2} = \tfrac{1}{\sqrt{2}}(U_1 \pm \mathrm{i}U_2); \tag{7}$$

then, we get

$$[L_\pm, Q_q] = \sqrt{2(2+1) - q(q \pm 1)}\, Q_{q\pm 1}, \qquad [L_0, Q_q] = qQ_q. \tag{8}$$

So the operators are 2-rank tensor operators.

Table 3. The anticommutation relations between $\mathcal{L}$'s.

$\{,\}$	L_x	L_y	L_z
L_x	$\frac{4}{3}I - \frac{\sqrt{3}}{3}Q_0 + U_1$	U_2	V_1
L_y		$\frac{4}{3}I - \frac{\sqrt{3}}{3}Q_0 - U_1$	V_2
L_z			$\frac{4}{3}I + \frac{2\sqrt{3}}{3}Q_0$

Table 4. The anticommutation relations between $\mathcal{Q}$'s.

$\{,\}$	U_1	U_2	V_1	V_2	Q_0
U_1	$\frac{4}{3}I + \frac{2\sqrt{3}}{3}Q_0$	0	$-V_1$	V_2	$\frac{2\sqrt{3}}{3}U_1$
U_2		$\frac{4}{3}I + \frac{2\sqrt{3}}{3}Q_0$	$-V_2$	$-V_1$	$\frac{2\sqrt{3}}{3}U_2$
V_1			$\frac{4}{3}I - \frac{\sqrt{3}}{3}Q_0 - U_1$	$-U_2$	$-\frac{\sqrt{3}}{3}V_1$
V_2				$\frac{4}{3}I - \frac{\sqrt{3}}{3}Q_0 + U_1$	$-\frac{\sqrt{3}}{3}V_2$
Q_0					$\frac{4}{3}I - \frac{2\sqrt{3}}{3}Q_0$

As we know, apart from magnetic moment, the spin-1 atomic nucleus has electric quadrupole moment. The magnetic moment and electric quadrupole moment of deuteron are $0.857\mu_N$ and $0.286 \times 10^{-2}b$, respectively. The Hamiltonian of a quadrupolar nucleus partially oriented in a liquid crystalline matrix, in the presence of a large magnetic field and having a first-order quadrupolar coupling [32], can be written as

$$H_1 = -\omega_0 I_z + \lambda\left(3I_z^2 - I^2\right) = -\omega_0 I_z + \sqrt{3}\lambda Q_0. \tag{9}$$

The general quadrupolar interaction of two spin-1 nuclei can be written as

$$\begin{aligned} H_2 &= \mathcal{K}_0 Q_0 \otimes Q_0 - \mathcal{K}_1(Q_1 \otimes Q_{-1} + Q_{-1} \otimes Q_1) + \mathcal{K}_2(Q_2 \otimes Q_{-2} + Q_{-2} \otimes Q_2) \\ &= \mathcal{K}_0 Q_0 \otimes Q_0 + \mathcal{K}_1(V_1 \otimes V_1 + V_2 \otimes V_2) + \mathcal{K}_2(U_1 \otimes U_1 + U_2 \otimes U_2). \end{aligned} \tag{10}$$

The Hamiltonian of the bilinear–biquadratic Heisenberg interaction can be written as

$$\begin{aligned} H_3 &= -\mathcal{K}(L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z) + \mathcal{K}'(L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z)^2 \\ &= -\left(\mathcal{K} + \tfrac{1}{2}\mathcal{K}'\right)(L_x \otimes L_x + L_y \otimes L_y + L_z \otimes L_z) \\ &\quad + \tfrac{1}{2}\mathcal{K}'\left[Q_0 \otimes Q_0 + (V_1 \otimes V_1 + V_2 \otimes V_2) + (U_1 \otimes U_1 + U_2 \otimes U_2) + \tfrac{8}{3}I\right]. \end{aligned} \tag{11}$$

From the commutation relations, we can easily see that the i$\mathcal{L}$'s span a subspace which constitutes a subalgebra; the other six bases span the complement space of the subalgebra. The subalgebra is isomorphism to $so(3)$ so we get an AI-type Cartan decomposition of $u(3)$. The Cartan subalgebra of the decomposition is span$\{\mathrm{i}Q_0, \mathrm{i}U_1, \mathrm{i}I_3\}$. A transform matrix

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -\mathrm{i} & 0 & -\mathrm{i} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{12}$$

can be used to transfer $\exp(\mathrm{i}\mathcal{L})$ to the conventional $SO(3)$ matrix by conjugation transformation.

The anticommutation relations are summarized in tables 3–5.

We have

$$\{\mathcal{L}, \mathcal{L}\} = \mathcal{Q}, \qquad \{\mathcal{L}, \mathcal{Q}\} = \mathcal{L}, \qquad \{\mathcal{Q}, \mathcal{Q}\} = \mathcal{Q}. \tag{13}$$

Table 5. The anticommutation relations between $\mathcal{L}$ and $\mathcal{Q}$.

$\{,\}$	U_1	U_2	V_1	V_2	Q_0
L_x	L_x	L_y	L_z	0	$-\frac{\sqrt{3}}{3}L_x$
L_y	$-L_y$	L_x	0	L_z	$-\frac{\sqrt{3}}{3}L_y$
L_z	0	0	L_x	L_y	$\frac{2\sqrt{3}}{3}L_z$

4. Dipole–quadrupole decomposition of two coupled spin-1 systems

The matrices of the gate acting on two coupled spin-1 systems are elements of the $U(9)$ group. We denote the bases of the Lie algebra $u(9)$ by the following expression:

$$F = \mathrm{i}(F^1 \otimes F^2), \qquad (F^j(j = 1, 2) \in \mathcal{L}, \mathcal{Q} \text{ or } I). \tag{14}$$

We carry out a decomposition of the Lie algebra $u(9)$ as follows:

$$u(9) = l \oplus p, \tag{15}$$

$$l := \mathrm{span}\{\mathrm{i}\mathcal{L} \otimes \mathcal{Q}, \mathrm{i}\mathcal{Q} \otimes \mathcal{L}, \mathrm{i}\mathcal{L} \otimes I, \mathrm{i}I \otimes \mathcal{L}\}, \tag{16}$$

$$p := \mathrm{span}\{\mathrm{i}\mathcal{L} \otimes \mathcal{L}, \mathrm{i}\mathcal{Q} \otimes \mathcal{Q}, \mathrm{i}\mathcal{Q} \otimes I, \mathrm{i}I \otimes \mathcal{Q}, \mathrm{i}I \otimes I\}. \tag{17}$$

Using the commutation and the anticommutation relations in the former section and the formula

$$[A \otimes B, C \otimes D] = \tfrac{1}{2}(\{A, C\} \otimes [B, D] + [A, C] \otimes \{B, D\}), \tag{18}$$

it is easy to verify that the decomposition of $u(9)$ is a Cartan decomposition. The subalgebra l is isomorphic to $so(9)$ and p is isomorphic to $so(9)^{\perp}$, so the Cartan decomposition of the system is of AI type whose rank is 9. The transform matrix $T' = T_1 \otimes T_1$ can be used to transfer e^l to the conventional $SO(9)$ matrix. The Cartan subalgebra is given by

$$\begin{aligned} a := \mathrm{span}\{&\mathrm{i}Q_0 \otimes Q_0, \mathrm{i}Q_0 \otimes I, \mathrm{i}I \otimes Q_0, \mathrm{i}U_1 \otimes U_1, \mathrm{i}U_1 \otimes I,\\ &\mathrm{i}I \otimes U_1, \mathrm{i}Q_0 \otimes U_1, \mathrm{i}U_1 \otimes Q_0, \mathrm{i}I \otimes I\}. \end{aligned} \tag{19}$$

Another equivalent Cartan subalgebra can be chosen as

$$\begin{aligned} \tilde{a} := \mathrm{span}\{&\mathrm{i}L_z \otimes L_z, \mathrm{i}Q_0 \otimes Q_0, \mathrm{i}Q_0 \otimes I, \mathrm{i}I \otimes Q_0, \mathrm{i}U_1 \otimes U_1, \mathrm{i}U_2 \otimes U_2,\\ &\mathrm{i}U_1 \otimes (I - \sqrt{3}Q_0), \mathrm{i}(I - \sqrt{3}Q_0) \otimes U_1, iI \otimes I\}. \end{aligned} \tag{20}$$

The Cartan subalgebra a contains the Q_0Q_0 interaction, and the $\tilde{a}$ contains the Ising term as well as the Q_0Q_0 interaction.

The second step of the decomposition is the decomposition of the Lie subalgebra l, that is

$$l = l' \oplus p', \tag{21}$$

with

$$\begin{aligned} l' := \mathrm{span}\{&\mathrm{i}L_z \otimes Q_0, \mathrm{i}L_z \otimes U, \mathrm{i}Q_0 \otimes L_z, \mathrm{i}U \otimes L_z,\\ &\mathrm{i}L_z \otimes I, \mathrm{i}I \otimes L_z, \mathrm{i}L_x \otimes V, \mathrm{i}L_y \otimes V, \mathrm{i}V \otimes L_x, iV \otimes L_y\}, \end{aligned} \tag{22}$$

$$\begin{aligned} p' := \mathrm{span}\{&\mathrm{i}L_z \otimes V, \mathrm{i}V \otimes L_z, \mathrm{i}L_x \otimes Q_0, \mathrm{i}L_x \otimes U, \mathrm{i}L_y \otimes Q_0, \mathrm{i}L_y \otimes U, iQ_0 \otimes L_x,\\ &\mathrm{i}U \otimes L_x, \mathrm{i}Q_0 \otimes L_y, \mathrm{i}U \otimes L_y, \mathrm{i}L_x \otimes I, \mathrm{i}L_y \otimes I, \mathrm{i}I \otimes L_x, \mathrm{i}I \otimes L_y\}. \end{aligned} \tag{23}$$

The l' is conjugate to $so(5) \oplus so(4)$, and p' is conjugate to $(so(5) \oplus so(4))^{\perp}$, so the Cartan decomposition of the system is of BDI type whose rank is 4. Its Cartan subalgebra is given by

$$a' := \text{span}\{\text{i}L_y \otimes I, \text{i}I \otimes L_y, \text{i}L_y \otimes (Q_0 + \sqrt{3}U_1), \text{i}(Q_0 + \sqrt{3}U_1) \otimes L_y\}. \tag{24}$$

Likewise for $K \in e^{l'}$, using the transformation matrices

$$R = \begin{pmatrix} I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \end{pmatrix}, \qquad T = \text{diag}\left\{ I_2, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, I_4 \right\}, \tag{25}$$

and $T' = T_1 \otimes T_1$, we can get $\tilde{K} = RTT'KT'^{\dagger}T^T R^T \in SO(5) \oplus SO(4)$.

We further factorize the subalgebra l' as

$$l' = l'' \oplus p'', \tag{26}$$

$$l'' := \text{span}\{\text{i}L_z \otimes Q_0, \text{i}L_z \otimes U, \text{i}Q_0 \otimes L_z, \text{i}U \otimes L_z, \text{i}L_z \otimes I, \text{i}I \otimes L_z\}, \tag{27}$$

$$p'' := \text{span}\{\text{i}L_x \otimes V, \text{i}L_y \otimes V, \text{i}V \otimes L_x, \text{i}V \otimes L_y\}. \tag{28}$$

The l'' is conjugate to $so(4) \oplus so(1) \oplus so(2) \oplus so(2)$ and p'' is conjugate to $(so(4) \oplus so(1) \oplus so(2) \oplus so(2))^{\perp}$. The Cartan subalgebra is given by

$$a'' := \text{span}\{\text{i}L_x \otimes V_2, \text{i}V_2 \otimes L_x, \text{i}(L_y \otimes V_1 - V_1 \otimes L_y)\}. \tag{29}$$

The Lie algebra l'' has two subalgebras which are expressed as

$$l''_1 := \text{span}\{\text{i}L_z \otimes Q_0, \text{i}L_z \otimes U, \text{i}Q_0 \otimes L_z, \text{i}U \otimes L_z\}, \tag{30}$$

$$l''_2 := \text{span}\{\text{i}L_z \otimes I, \text{i}I \otimes L_z\}, \tag{31}$$

with l''_1 conjugate to the Lie algebra $so(4)$ and l''_2 conjugate to the Lie algebra $so(2) \oplus so(2)$.

5. The realization of the two-qutrit gate

The key of implementing the quantum logic gate is to decompose the unitary matrix into the product of realizable matrices. Based on the previous discussion, the decomposition of the $U(9)$ matrix for the two-qutrit gate is given by

$$X = K_1 A''_1 K_2 A'_1 K_3 A''_2 K_4 A K_5 A''_3 K_6 A'_2 K_7 A''_4 K_8, \tag{32}$$

where A, A'_j ($j = 1, 2$) and A''_j ($j = 1, 2, 3, 4$) belong to the Abel groups associated with Cartan subalgebras that appear in each decomposition step. K_j ($j = 1, 2, \ldots, 8$) are conjugate to the Lie group of $SO(4) \oplus SO(1) \oplus SO(2) \oplus SO(2)$. The $SO(4)$ group is isomorphic to $SO(3) \oplus SO(3)$, and the well-known Euler decomposition can be used to decompose the $SO(3)$ matrix. Hence the $U(9)$ matrix for the two-qutrit gate can be factorized into the product of single parameter subgroups. The Hamiltonian of the system can be written as

$$H = H_d + \sum_i \nu_i(t) H_i, \tag{33}$$

where H_d is the part of Hamiltonian that is internal to the system and we call it the free evolution or drift Hamiltonian and $\sum_i \nu_i(t) H_i$ is the part of Hamiltonian that can be externally changed and we call it the control Hamiltonian. The equation (32) can be written in an exponential form by the correspondence between the Lie group and Lie algebra, so as to make them easily relate to the free Hamiltonian and control field of the system. Each factor can be realized by the control processes, or by the drift processes and some suitable transformations [21, 33].

In order to illustrate dipole–quadrupole decomposition of spin 1's we have discussed above, we take the ternary SWAP gate as an example as in [16]. The ternary SWAP gate is defined as

$$X_{sw} : |i\rangle_1 \otimes |j\rangle_2 \rightarrow |j\rangle_1 \otimes |i\rangle_2, \tag{34}$$

where $i, j = 0, 1, 2$, and $\{|0\rangle, |1\rangle, |2\rangle\}_{1,2}$ are orthonormal bases for the Hilbert space of the subsystems. The matrix representation of this gate is given by

$$X_{sw} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

Here $X_{sw} \in O(9) \subset U(9)$, and therefore the decomposition of $U(9)/SO(9)$ can be simplified to $X_{sw} = X \cdot A$, where A is a diagonal matrix whose determinant is -1. Taking the transformation $\tilde{X} = RTT'XT'^{\dagger}T^T R^T$, we find that the $\tilde{X}$ belongs to the $SO(5) \oplus SO(4)$ group; the second step of the decomposition can be omitted. The further decomposition can be carried out as described above. The result is not unique, and one of the results is that

$$X_{sw} = L_1 L_2 L_3 L_4 A'' A = e^{l_1} e^{l_2} e^{l_3} e^{l_4} e^{a''} e^{a}, \tag{36}$$

where

$$l_1 = -\frac{\pi}{4}(\mathrm{i}L_z \otimes U_2), \tag{37}$$

$$l_2 = \pi(\mathrm{i}I \otimes L_z), \tag{38}$$

$$l_3 = \frac{\pi}{2}\mathrm{i}(L_z \otimes U_1 + U_1 \otimes L_z), \tag{39}$$

$$l_4 = \frac{\pi}{4}(\mathrm{i}U_2 \otimes L_z), \tag{40}$$

$$a'' = \frac{\pi}{4}[\mathrm{i}3L_x \otimes V_2 + \mathrm{i}V_2 \otimes L_x - \mathrm{i}(L_y \otimes V_1 - V_1 \otimes L_y)], \tag{41}$$

$$a = \pi(\mathrm{i}I \otimes I + \mathrm{i}L_z \otimes L_z), \quad \text{or} \quad a = \pi(\mathrm{i}I \otimes I - \mathrm{i}3Q_0 \otimes Q_0). \tag{42}$$

From the suitable transformation, they can be transferred to the realizable matrices of the system. If the system coupled with the Ising interaction and suitable control processes are used, to implement the gate it needs 9 drift processes and 25 basic control processes [33].

6. Conclusion

Using the angular momentum operators and quadrupole moment operators as bases of $su(3)$, we investigate a decomposition of the matrix for two coupled spin-1 systems. By successive Cartan decompositions, the unitary matrix can be decomposed into product of one-parameter Lie subgroups. The method can be used to investigate the realization of two-qutrit logic gates.

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